Majorana Depolarization of Hydrogen, Deuterium, or Tritium Atoms
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Majorana Depolarization of Hydrogen, Deuterium, or Tritium Atoms

by

Gerald G. Ohlsen
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ABSTRACT
The theory required to follow the behavior of a hydrogen, deuterium, or tritium atom in a time-dependent magnetic field is described. A computer code is included, and some numerical results of interest to the design of Lamb-shift polarized-ion sources are presented. A brief discussion of depolarization effects in pick-up or stripping of two electrons is also presented.

1. INTRODUCTION
In all sources of polarized hydrogen or deuterium ions which have been proposed, the process involves, first, the production of a beam of atoms with a net nuclear polarization and, second, the ionization of these polarized atoms. It is frequently required to change the magnetic field strength or direction or both from one value to another between the point at which the polarized atoms are produced and that at which they are ionized. Thus, one is required, in the design of such devices, to estimate depolarization effects caused by unwanted transitions between the various hyperfine states when atoms are subjected to time-varying magnetic fields. (Such transitions are usually referred to as Majorana transitions.) In most cases one wishes to design magnetic field shapes in a way which eliminates or reduces these effects. Problems of this type are particularly important in "Lamb-shift" sources, where the atomic beam velocities are large (~30 cm/usec).

A related problem, which we will also consider, is the "zero field crossing" technique of polarization enhancement. In this method it is required to reverse the direction of the magnetic field in a way such that a certain transition is made with high probability while the remaining transitions occur with low probability. We will also briefly discuss the depolarization of ions that may occur when two electrons are stripped or picked up, as in a tandem accelerator stripper or a cesium adding canal.

2. THEORY
The Schroedinger equation for a one-electron atom may be written

$$\frac{\partial\Psi}{\partial t} = H_0 \Psi, \quad (1)$$

where $H_0$ is that part of the Hamiltonian which does not depend on electronic or nuclear spin and

$$H_0 = \mu_o g_e \mathbf{j} + \mu_n g_I \mathbf{I} + \frac{\mathbf{B}}{c} + \alpha \mathbf{I} \cdot \mathbf{J}. \quad (2)$$

In the above expression $\mu_o = e\hbar/2mc$ (the Bohr magneton); $\mu_n = e\hbar/2mc$ (the nuclear magneton); $g_e$ and $g_I$ are the electronic and nuclear g-factors, respectively; and $\alpha$ is related as follows to the zero field hyperfine energy separation ($\Delta W$):

$$\alpha = \Delta W/I(2I + 1) \text{ if } I < J$$
$$\alpha = \Delta W/J(2J + 1) \text{ if } J < I. \quad (3)$$

For hydrogen atoms $I = J = 1/2$ and thus $\alpha = \Delta W/2$, deuterium atoms $I = 1, J = 1/2$, and therefore $\alpha = \Delta W/3$. 

3
If we assume a complete set of functions \( u_n \) which satisfy \( H u_n = E_n u_n \), we may write the general wave function as

\[
y = \sum_n E_n u_n e^{-iE_n t/\hbar}.
\]

We consider the \( u_n \) to be the four strong field states \( \psi(m, m_J) \) for hydrogen atoms (six states for deuterium atoms), where the quantization axis is specified and stationary and where \( m_I \) and \( m_J \) are the nuclear and electronic magnetic quantum numbers, respectively. All of these states have the same space wave function and, hence, the same eigenvalue of the operator \( H_0 \). Substituting Eq. 4 into Eq. 1, multiplying from the left by \( u_m^* \), and integrating over the space variables, we obtain the equations of motion of the probability amplitudes:

\[
\hbar \dot{\psi}_m = \sum_n E_n \psi_m^* u_n \psi_m.
\]

If we put explicit values of the matrix elements into the above expression and use the quantum numbers \( m_I \) and \( m_J \) to label the states, Eq. 5 becomes

\[
\hbar \dot{\psi}_{m_I, m_J} = \left( \psi_{m_I, m_J} + \alpha \psi_{m_I, m_J} \right) b_{m_I, m_J}.
\]

where \( \alpha \) is the nuclear and \( g_J \) is the electronic g-factor (see Table I). We define the following parameters:

\[
k = \frac{\alpha}{1836.1 g_J},
\]

\[
B_o = \frac{\alpha}{\hbar \psi_J}
\]

With these definitions Eq. 6 reduces, for hydrogen or tritium atoms, to the following four coupled differential equations:

\[
\begin{bmatrix}
\dot{\psi}_1 \\
\dot{\psi}_2 \\
\dot{\psi}_3 \\
\dot{\psi}_4
\end{bmatrix}
= \begin{bmatrix}
\frac{\psi_{m_I, m_J}}{\hbar \psi_m} & \frac{(1-k)B_z + kB_o}{\hbar \psi_m} & \frac{kB_o}{\hbar \psi_m} & 0 \\
\frac{kB_o}{\hbar \psi_m} & \frac{(1-k)B_z - kB_o}{\hbar \psi_m} & \frac{\psi_{m_I, m_J}}{\hbar \psi_m} & 0 \\
0 & \frac{kB_o}{\hbar \psi_m} & \frac{(1-k)B_z + kB_o}{\hbar \psi_m} & \frac{\psi_{m_I, m_J}}{\hbar \psi_m} \\
\frac{kB_o}{\hbar \psi_m} & \frac{(1-k)B_z - kB_o}{\hbar \psi_m} & \frac{\psi_{m_I, m_J}}{\hbar \psi_m} & 0
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4
\end{bmatrix}.
\]

**TABLE I**

**HYPERFINE STRUCTURE PARAMETERS**

<table>
<thead>
<tr>
<th>State</th>
<th>( g_I )</th>
<th>( g_J )</th>
<th>( e )</th>
<th>( B_z )</th>
<th>( \Delta W )</th>
<th>( k )</th>
<th>( B_o )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1S Hydrogen</td>
<td>5.585486</td>
<td>2.00229</td>
<td>1.522x10^{-3}</td>
<td>507.991</td>
<td>10420.406</td>
<td>1.520x10^{-3}</td>
<td>506.820</td>
</tr>
<tr>
<td>2S Hydrogen</td>
<td>5.585486</td>
<td>2.00229</td>
<td>1.522x10^{-3}</td>
<td>63.450</td>
<td>177.557</td>
<td>1.520x10^{-3}</td>
<td>63.354</td>
</tr>
<tr>
<td>1S Deuterium</td>
<td>0.857407</td>
<td>2.00229</td>
<td>0.233x10^{-3}</td>
<td>116.842</td>
<td>327.384</td>
<td>0.233x10^{-3}</td>
<td>77.877</td>
</tr>
<tr>
<td>2S Deuterium</td>
<td>0.857407</td>
<td>2.00229</td>
<td>0.233x10^{-3}</td>
<td>14.605</td>
<td>40.924</td>
<td>0.233x10^{-3}</td>
<td>9.735</td>
</tr>
<tr>
<td>1S Tritium</td>
<td>5.957680</td>
<td>2.00229</td>
<td>1.623x10^{-3}</td>
<td>542.059</td>
<td>1516.702</td>
<td>1.620x10^{-3}</td>
<td>541.181</td>
</tr>
<tr>
<td>2S Tritium</td>
<td>5.957680</td>
<td>2.00229</td>
<td>1.623x10^{-3}</td>
<td>67.759</td>
<td>189.594</td>
<td>1.620x10^{-3}</td>
<td>67.650</td>
</tr>
</tbody>
</table>
where $B_x$, $B_y$, and $B_z$ are the components of the applied $B$ field (arbitrary time variation), $B_+ = B_x + iB_y$, $B_- = B_x - iB_y$, and states 1-4 are the strong field states $\psi(m_I, m_J)$ ordered according to their energy in a magnetic field (as indicated in Fig. 1). For deuterium atoms Eq. 6 reduces to six coupled differential equations:

$$
\begin{align*}
E_1 &= (1+\kappa)B_z+B_0 \quad kB_+ \quad 0 \\
E_2 &= kB_- \quad kB_+ \\
E_3 &= \kappa \quad kB_- \\
E_4 &= 0 \quad 0 \quad kB_- \\
E_5 &= 0 \quad kB_- \quad B_0/\sqrt{2} \\
E_6 &= kB_- \quad B_0/\sqrt{2} \quad 0
\end{align*}
$$

where the states 1-6 are the strong field states ordered as described above (as indicated in Fig. 2). The parameter $\kappa$ is a small number ($<10^{-5}$) and thus could be neglected for most purposes, although this assumption has not been made in the numerical calculations to be described in subsequent sections.

We note in passing that the parameters $\kappa$ and $B_0$ are different from those which arise in the description of the energy levels of the atom. Specifically, the "Breit-Rabi formula" for hydrogen or tritium atoms is

$$
W = \frac{1}{2} \alpha \beta \gamma^2 (1+2\mu \gamma + \gamma^2)^2 + c \Delta \gamma \mu \gamma ,
$$

where $\mu = \mu_L + \mu_J$, and where the plus sign applies to states 1, 2, and 3 while the minus sign applies to state 4. The quantity $\gamma = B/B_0$, with $B_0 = \Delta \gamma /[(1-k)\mu, \mu_J]$ and $c = k/(1-k)$. For deuterium atoms we have

$$
W = \frac{1}{6} \alpha \beta \gamma^3 [1+3(1+2\delta \gamma + \gamma^2)^2 + c \Delta \gamma \mu \gamma .
$$

The definitions of $\gamma$ and of $c$ remain the same, but in this case we have $B_0 = 3B_0 /[(1-k)]$. The plus sign applies to states 1-4 while the minus sign applies to states 5-6. The numerical values of the parameters used in Eqs. 8-11 are given in Table I.

The solution of the above differential equa-
tions (Eqs. 8 or 9) for the field variation of interest, then, is a straightforward computational problem. However, the states we have used in the description are the strong field states with respect to a fixed z-axis. Since, in general, the direction of the magnetic field varies, it is somewhat easier to interpret the results if the coordinate system is rotated, so that the new z-axis lies along the (time-varying) magnetic field direction. The rotation transformation for composite states may be derived from the spin-1/2 and spin-1 rotation matrices. For a spin-1/2 particle, the rotation transformation may be represented by the matrix equation

\[
\begin{bmatrix}
  b_+ \\
  b_-
\end{bmatrix} =
\begin{bmatrix}
  a_{++} & a_{+-} \\
  a_{-+} & a_{--}
\end{bmatrix}
\begin{bmatrix}
  b_+ \\
  b_-
\end{bmatrix},
\]

where the subscript refers to the sign of the magnetic quantum number \(m(\pm 1/2)\). Explicitly, the coefficients are

\[
\begin{align*}
  a_{++} &= \cos \left(\frac{\beta}{2}\right) \exp \left[-i\gamma(a+\gamma)\right] \\
  a_{--} &= \cos \left(\frac{\beta}{2}\right) \exp \left[i\gamma(a-\gamma)\right] \\
  a_{-+} &= -\sin \left(\frac{\beta}{2}\right) \exp \left[-i\gamma(a-\gamma)\right] \\
  a_{+-} &= \sin \left(\frac{\beta}{2}\right) \exp \left[i\gamma(a+\gamma)\right],
\end{align*}
\]

where \(a, \beta, \text{ and } \gamma\) are the positive Euler angles\(^2\) which rotate the initial coordinate system (unprimed) into the final coordinate system (primed). For hydrogen or tritium atoms (in strong field representation), the rotation matrix is essentially a direct product of the electronic and nuclear rotation. The complete rotation matrix may be written

\[
\begin{bmatrix}
  b_{1}' \\
  b_{2}' \\
  b_{3}' \\
  b_{4}'
\end{bmatrix} =
\begin{bmatrix}
  a_{++}c_{++} & a_{++}c_{+0} & a_{++}c_{+-} & a_{--}c_{++} \\
  a_{++}c_{+0} & a_{++}c_{+0} & a_{++}c_{+0} & a_{--}c_{+0} \\
  a_{++}c_{+-} & a_{++}c_{+0} & a_{++}c_{+0} & a_{--}c_{+-} \\
  a_{--}c_{++} & a_{--}c_{+0} & a_{--}c_{+0} & a_{--}c_{++}
\end{bmatrix}
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4
\end{bmatrix},
\]

where states 1-4 are ordered as in Fig. 1. (Note that the states 3 and 4 are interchanged in relation to the notation one would choose if a true direct product notation were to be applicable.) For a spin-1 particle, the rotation matrix is

\[
\begin{bmatrix}
  b_+ \\
  b_-
\end{bmatrix} =
\begin{bmatrix}
  c_{++} & c_{+0} & c_{+-} \\
  c_{+0} & c_{+0} & c_{+0} \\
  c_{+-} & c_{-0} & c_{--}
\end{bmatrix}
\begin{bmatrix}
  b_+ \\
  b_-
\end{bmatrix},
\]

in an obvious notation. In this case the coefficients are given by

\[
\begin{align*}
  c_{++} &= (\sin \frac{\beta}{2}) \cos \frac{\gamma}{2} \exp \left[-i(a+\gamma)\right] \\
  c_{+0} &= (\sin \frac{\beta}{2}) \sin \frac{\gamma}{2} \exp \left[i(a-\gamma)\right] \\
  c_{+-} &= (\sin \frac{\beta}{2}) \cos \frac{\gamma}{2} \exp \left[-i(a-\gamma)\right] \\
  c_{--} &= (\sin \frac{\beta}{2}) \sin \frac{\gamma}{2} \exp \left[i(a+\gamma)\right].
\end{align*}
\]

Finally, it is usually more convenient to describe the final system in terms of eigenstates of the particular magnetic field strength rather than in terms of the strong field eigenstates. If the fields are changed slowly enough, the system will remain in a particular state; that is, the energy of the system will remain on one of the lines 1-4 for hydrogen atoms or 1-6 for deuterium atoms (provided that, initially, the system was in such an eigenstate). These eigenstates are, in general, linear combinations of the strong field basis states used in Eqs. 8 and 9. In terms of these "intermediate
field states, failures of adiabaticity will appear in the form of transitions from the initial state (or the state to which it would have been transformed if the process were adiabatic) to one or more of the other states. In terms of the strong field state amplitudes (primed), the amplitudes of the hydrogen or tritium atom intermediate-field eigenstates (double-primed) may be written, for arbitrary $B$, as

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \sqrt{2}(1+\delta) & 0 & \sqrt{2}(1-\delta) \\
0 & 0 & 1 & 0 \\
0 & -\sqrt{2}(1-\delta) & 0 & \sqrt{2}(1+\delta)
\end{bmatrix}
\begin{bmatrix}
b_1' \\
b_2' \\
b_3' \\
b_4'
\end{bmatrix}
= 
\begin{bmatrix}
b_1'' \\
b_2'' \\
b_3'' \\
b_4''
\end{bmatrix},
$$

(18)

where $\delta = x/(1 + x^2)^{1/2}$ and $x = B/B_1$. For deuterium atoms we have

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \sqrt{2}(1+\delta_+) & 0 & \sqrt{2}(1-\delta_-) \\
0 & 0 & \sqrt{2}(1+\delta_-) & 0 \\
0 & 0 & 0 & 1 \\
0 & -\sqrt{2}(1-\delta_-) & 0 & \sqrt{2}(1+\delta_-)
\end{bmatrix}
\begin{bmatrix}
b_1' \\
b_2' \\
b_3' \\
b_4' \\
b_5' \\
b_6'
\end{bmatrix}
= 
\begin{bmatrix}
b_1'' \\
b_2'' \\
b_3'' \\
b_4'' \\
b_5'' \\
b_6''
\end{bmatrix},
$$

(19)

where

$$
\delta_+ = (x+1)/(1+2x/3+x^2)^{1/2}
$$

$$
\delta_- = (x-1)/(1-2x/3+x^2)^{1/2}
$$

and again $x = B/B_1$. From these eigenfunctions one can easily calculate the nuclear and electronic polarization of a beam whose atoms are in a particular pure state (see Table II).

The nuclear polarization parameters may be written in terms of the strong field amplitudes by means of the appropriate projection operators. For reference we write the expressions for the quantities of principal interest:

a) Hydrogen or Tritium Atoms

$$
\langle S_z \rangle = |b_1|^2 + |b_4|^2 - |b_2|^2 - |b_3|^2
$$

$$
\langle S_x \rangle = 2 \text{Re}(b_1^* b_2^* b_4^* b_3)
$$

$$
\langle S_y \rangle = 2 \text{Im}(b_1^* b_2^* b_4^* b_3)
$$

(20)

b) Deuterium Atoms

$$
\langle S_z \rangle = |b_1|^2 + |b_4|^2 - |b_2|^2 - |b_3|^2
$$

$$
\langle S_x \rangle = 2 \text{Re}(b_1^* b_2^* b_3^* b_5^* b_6^* b_5^*)
$$

$$
\langle S_y \rangle = 2 \text{Im}(b_1^* b_2^* b_3^* b_5^* b_6^* b_5^*)
$$

(21)

where all expectation values refer to nuclear polarization. Similar expressions may be written for electronic polarization. It is clear that we may use these expressions to calculate the polarization parameters in either the initial coordinate system (unprimed amplitudes), or in the rotated coordinate system (primed amplitudes), depending on the desired reference axes.

| TABLE II |
| POLARIZATION IN INTERMEDIATE FIELDS |
| Hydrogen or Tritium Atoms |
| State | $P$ (nuclear) | $P$ (electronic) |
| 1 | 1 | 1 |
| 2 | -6 | 6 |
| 3 | -1 | 1 |
| 4 | 6 | -6 |

| Deuterium Atoms |
| State | $P_3$ (nuclear) | $P_{33}$ (nuclear) | $P$ (electronic) |
| 1 | 1 | 1 | 1 |
| 2 | $b_4^*(1-\delta_+)$ | $-b_4^*(1+3\delta_+)$ | $\delta_+$ |
| 3 | $-b_4^*(1+\delta_-)$ | $-b_4^*(1-3\delta_-)$ | $\delta_-$ |
| 4 | -1 | 1 | -1 |
| 5 | $-b_4^*(1-\delta_-)$ | $-b_4^*(1+3\delta_-)$ | $-\delta_-$ |
| 6 | $b_4^*(1+\delta_+)$ | $-b_4^*(1-3\delta_+)$ | $-\delta_+$ |
The equations of motion of the probability amplitudes (Eqs. 8 and 9) may be solved in a straightforward way if the field is assumed to be constant and if the (negligible) nuclear terms are omitted. We present these solutions here primarily because they are useful in understanding and describing the nature of the more general solutions.

If we neglect the nuclear term in the Hamiltonian, the differential equations for the hydrogen or tritium atom become

\[
\begin{align*}
\dot{\psi}_1 &= \frac{1}{2}(x+1)\omega \psi_1 \\
\dot{\psi}_2 &= \frac{1}{2}(x-1)\omega \psi_2 + \psi_3 \\
\dot{\psi}_3 &= \frac{1}{2}(x+1)\omega \psi_2 \\
\dot{\psi}_4 &= \frac{1}{2}(x+1)\omega \psi_4 ,
\end{align*}
\]

where the z axis is defined by the (constant) magnetic field direction, \(x = B/B_0\), and \(\omega = \Delta \omega/\hbar\). If the initial conditions are \(\psi_1 = \psi_1, \psi_2 = \psi_2, \psi_3 = \psi_3, \) and \(\psi_4 = \psi_4,\) the general solution may be written

\[
\begin{align*}
\psi_1 &= \psi_1 \exp\left[-i(\psi+2)\psi t\right] \\
\psi_2 &= \psi_2 \exp\left[i(\psi+2)\psi t\right]
\end{align*}
\]

\[
\begin{align*}
\dot{\psi}_2 &= \frac{1}{2}\left[\psi_2(1-x^2)-\psi_4(1)\right]\exp[i(\psi+2)\psi t] \\
\dot{\psi}_3 &= \frac{1}{2}\left[\psi_2(1-x^2)+\psi_4(1)\right]\exp[i(\psi+2)\psi t] \\
\dot{\psi}_4 &= \frac{1}{2}\left[\psi_2(1-x^2)+\psi_4(1)\right]\exp[i(\psi+2)\psi t] \\
\dot{\psi}_5 &= \frac{1}{2}\left[\psi_2(1-x^2)-\psi_4(1)\right]\exp[i(\psi+2)\psi t] \\
\dot{\psi}_6 &= \frac{1}{2}\left[\psi_2(1-x^2)+\psi_4(1)\right]\exp[i(\psi+2)\psi t] ,
\end{align*}
\]

where \(B = \sqrt{1+x^2}\). Only values of \(\psi_1 - \psi_4\) for which \(\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2 = 1\) correspond to physical initial states.

Again neglecting the nuclear term in the Hamiltonian, the differential equations for the deuterium atom become

\[
\begin{align*}
\dot{\psi}_1 &= \frac{1}{2}(x+1)\omega \psi_1 \\
\dot{\psi}_2 &= \frac{1}{2}x \omega \psi_2 + \frac{1}{\sqrt{2}} \omega \psi_6 \\
\dot{\psi}_3 &= \frac{1}{2}(x-1)\omega \psi_3 + \frac{1}{\sqrt{2}} \omega \psi_5 \\
\dot{\psi}_4 &= \frac{1}{2}(x+1)\omega \psi_4 \\
\dot{\psi}_5 &= \frac{1}{\sqrt{2}} \omega \psi_3 - \frac{1}{2}x \omega \psi_5 \\
\dot{\psi}_6 &= \frac{1}{\sqrt{2}} \omega \psi_2 - \frac{1}{2}(x+1)\omega \psi_6 ,
\end{align*}
\]

where \(x = B/B_0, \omega = 2\Delta \omega/3h,\) and again the z axis is defined by the magnetic field. The solutions may be written in terms of the initial vector \(\psi\) as follows:

\[
\begin{align*}
\dot{\psi}_1 &= \psi_1 \exp\left[-i(\psi+2)\psi t\right] \\
\dot{\psi}_2 &= \psi_2 \exp\left[i(\psi+2)\psi t\right]
\end{align*}
\]

\[
\begin{align*}
\dot{\psi}_2 &= \psi_2 \exp\left[i\psi(1-x^2)\psi t\right] \\
\dot{\psi}_3 &= \psi_3 \exp\left[i\psi(1-x^2)\psi t\right] \\
\dot{\psi}_4 &= \psi_4 \exp\left[i\psi(1-x^2)\psi t\right] \\
\dot{\psi}_5 &= \psi_5 \exp\left[i\psi(1-x^2)\psi t\right] \\
\dot{\psi}_6 &= \psi_6 \exp\left[i\psi(1-x^2)\psi t\right] ,
\end{align*}
\]

where \(\psi = \sqrt{x^2+2}/\psi\). Only those values of \(\psi_1 - \psi_6\) for which \(\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2 + \psi_5^2 + \psi_6^2 = 1\) correspond to physical initial states.

3. APPLICATIONS

We will consider several applications of the theory just described, all of which are of interest in the design and utilization of polarized-ion sources. For the most part, where numerical results are presented, we will have in mind polarized-ion sources of the Lamb-shift type. We will consider the following problems:

a) "Adiabatic reduction" of a large (longitudinal) to a small (longitudinal) magnetic field;

b) Adiabatic reduction of a large (longitudinal) to a small (transverse) magnetic field;

c) The sudden zero field crossing technique of polarization enhancement;

d) Depolarization effects associated with the addition of two electrons to a polarized H\(^+\) or D\(^+\) ion beam.

We first make some general observations about the conditions required for adiabaticity. At low fields a one-electron atom in a pure state will behave like an elementary particle which has the magnetic moment of the electron but the total spin angular momentum of the atom. Since a free electron precesses about a field at the rate of \(\mu_0 e/\hbar = 2.8 \text{ MHz/G}\), a hydrogen atom in the \(F = 1\) state...
will precess at a rate of 1.4 MHz/G. (The pure \( F = 0 \) state has no polarization; thus, its precession rate, which would be infinite from this point of view, has no physical interpretation.) A deuterium atom in the \( F = 3/2 \) state will precess at 0.93 MHz/G while one in an \( F = 1/2 \) state will precess at 2.8 MHz/G. Thus, for low fields, transitions will be induced only if the field direction changes rapidly with respect to the appropriate one of these precession frequencies. We will adopt the term "critical frequency" to denote the particular precession frequency which serves as the boundary between the zero transition (adiabatic) region and the complete transition (diabatic) region.

At high fields the critical frequency is directly related to the hyperfine splitting. To illustrate the connection, we consider a hydrogen atom which has, at zero time, its electron spin aligned with the magnetic field (+z-axis) and its proton spin aligned with the +x-axis. In terms of the solutions given in Section 2, the initial conditions which represent this situation are \( \epsilon_1 = \epsilon_2 = 1/\sqrt{2} \) and \( \epsilon_3 = \epsilon_4 = 0 \). From Eqs. 20 and 23, we find that

\[
\langle S_x \rangle = 2\Re(b_1^*b_2+b_2^*b_3) + \cos(\Delta W/2\hbar)t. \tag{26}
\]

Thus, if the field is sufficiently strong to maintain the alignment of the electron, i.e., for \( x \gg 1 \), the component of the proton spin angular momentum which is not parallel to \( B \) will precess around the magnetic field at one-half of the hyperfine frequency. The classical picture that is involved is as follows. Neglecting, as in Eqs. 22 and 24, the \( \nu_B \) term in the Hamiltonian, we may say that the electron precesses about the applied magnetic field with a frequency corresponding to the free-electron precession rate, while the nucleus precesses about the electron with a frequency closely related to the normal hyperfine splitting. If the external field is to be changed in an adiabatic manner, it must be changed slowly with respect to both frequencies. At low fields the critical frequency therefore approaches the electron precession frequency (modified by the total angular momentum of the atom) while at high fields it approaches one-half of the normal hyperfine frequency.

For both the low- and high-field regions, the critical frequencies arrived at, in this semiclassical picture, correspond exactly to the energy separation between the initial state and the nearest neighboring state. For intermediate field strengths the motion is complicated; the electron and proton may be said to "tumble" about each other. However, from the form of Eq. 26 it is clear that the separation between the initial and the adjacent state still corresponds to the critical frequency (see Fig. 3).

For deuterium atoms we obtain similar results. For an atom which has, at zero time, the electron aligned with the field (+z-axis) and the deuteron in an \( m_1 = 1 \) state with respect to the +x-axis, the appropriate initial conditions are \( \epsilon_1 = \epsilon_3 = 1/2 \), \( \epsilon_2 = 1/\sqrt{2} \), and \( \epsilon_4 = \epsilon_5 = \epsilon_6 = 0 \). From Eqs. 21 and 25, we find that

\[
\langle S_x \rangle = \sqrt{2} \Re(b_1^*b_2+b_2^*b_3+b_5^*b_4+b_4^*b_5) + \cos(\Delta W/3\hbar)t. \tag{27}
\]

In this case the high-field critical frequency is one-third of the hyperfine frequency. However, we note that the critical frequency again corresponds to the energy separation between the initial and the adjacent state (see Fig. 3).

Adiabatic Reduction of a Large (Longitudinal) to a Small (Longitudinal) Magnetic Field

Maxwell's equations imply the following (first order) relation between the radial and axial components of a cylindrically symmetric magnetic field:

\[
B_r = -\frac{\nu_B \nu_B B_z}{2/\delta}. \tag{28}
\]
Thus, except at \( r = 0 \), a changing magnetic field strength \( B_z \) is always accompanied by a radial field component. The angle between the field direction and the \( z \)-axis is therefore given by

\[
\tan \theta = \frac{B_r}{B_z} = -\frac{3B_z}{2B_z^2} \frac{\Delta t}{\Delta z}.
\] (29)

For example, if we have a uniformly falling field \( (B_z = -cz) \), Eq. 29 becomes

\[
\tan \theta = \frac{r}{2z} = \frac{r}{2vt},
\] (30)

where \( z = vt \) for a beam moving with velocity \( v \). The same result holds for an exponentially falling field \( (B_z = ce^{-z/2}) \). Thus, except at \( r = 0 \), for any kind of declining field, there will be a changing field direction; it is this rate of change that must be kept small with respect to the relevant critical precession frequency. For a linearly falling field, this angular rate of change is

\[
\omega = \frac{d}{dt} \left( \tan^{-1} \frac{r}{2vt} \right) = \frac{-(r/2v)}{(r/2v)^2 + t^2}.
\] (31)

Except at small \( t \) (and therefore small \( B \)) the rate of rotation is inversely proportional to \( r \). Thus, the outer region of a large beam will be less subject to depolarization than the inner region.

We now consider the depolarization effects for some particular field shapes. (These results were obtained with the computer code given in the Appendix.) Figure 4 presents some numerical results for a field which falls from an initial value of 575 G to a final value of 5 G with an exponential law:

\[
B_z = 575 e^{-z/2} + 5.
\] (The particular velocity of 30 cm/\( \mu \)sec, which is that used in a Lamb-shift polarized-ion source, is assumed for all numerical results presented in this report.) Particles traveling on the axis \( (r = 0) \) experience no change in field direction and hence undergo no transitions. In each case we plot the retained fraction of the atoms; that is, the fraction which does not make a transition to any other quantum state.

Figure 5 presents curves similar to those of Fig. 4 for a particular magnetic field configuration approximating that which may be obtained by a solenoid in an iron cylinder with a small oppositely directed correction current about one diameter from the main solenoid end. (The exact field shape used is shown in Fig. 6.) The depolarization effects for
a field of this general shape are somewhat larger than with the exponential field. This points out that the exponential shape is nearly ideal, since $B_z$ changes more and more slowly as the (more critical) lower field levels are reached.

Adiabatic Reduction of a Large (Longitudinal) to a Small (Transverse) Field

Figure 7 presents the retained fraction for an exponentially declining axial field and a transverse final field direction; that is, the field on the axis, in gauss, is described by $B_z = 5.75 e^{-z/2}$, $B_x = 5$. It is somewhat easier to induce depolarization in this case than in the case where the final field is longitudinal, because $\theta = \tan^{-1} B_x / B_z$ must change by 90°, whereas for a longitudinal final field, $\theta$ increases to some maximum value and then returns to zero. The critical region is where $B_x = B_z$, since the field rotation rate is maximum there. The maximum allowable $\omega$ is determined by the total magnetic field $B = \sqrt{B_x^2 + B_y^2 + B_z^2}$. The details of the shape of the $B_x$ field are apparently unimportant so long as $B_x$ has risen to its full value before $B_z$ drops below one or two times the final $B_x$ value.

Figure 8 shows the retained fraction for a final transverse field of 5 G ($B_x = 5$) and for a longitudinal field 5 G less than that plotted in Fig. 6. (The axial field approaches zero for large displacement.) Again the depolarization effects are more severe in this case than in the exponential one. This is because the rate of fall of the axial field in the critical region (near 5 G) is greater in the present case.

From Figs. 4, 5, 7, and 8, it is seen that, of the states considered, state-1 hydrogen atoms are depolarized the least, while state-2 deuterium atoms are depolarized the most. On the basis of the critical-frequency arguments, we expect hydrogen atoms to be less subject to depolarization than deuterium atoms (in agreement with the calculations). However, these arguments do not account for the difference between state-1 and state-2 deuterium.
Fig. 8. Retained fraction of hydrogen atoms in state 1 (I), deuterium atoms in state 1 (II), and deuterium atoms in state 2 (III) for a particular axial field shape (see text) and a transverse final field. The curves and the abscissa are labeled as in Fig. 5. A velocity of 30 cm/µsec is assumed.

Atoms. Possibly state-2 deuterium atoms are more readily depolarized than state-1 atoms because there are two nearby states instead of one to which transitions may occur.

A transverse field destroys cylindrical symmetry, so that particles at a given radius, but at different azimuthal angle with respect to the beam axis, will undergo different field rotations. Thus, atoms in different parts of a beam will be subject to different transition probabilities. This is, of course, not important if the transition probabilities are all kept near zero. The curves presented in Figs. 7 and 8 assume an average situation; i.e., they correspond to a ray in the y-z plane and to a final transverse field in the x-direction.

Sudden Zero Field Crossing Technique of Polarization Enhancement

It has been suggested by Sona that it should be possible to enhance the polarization of a metastable H or D beam with the aid of a sudden reversal of the magnetic field direction. In this scheme the magnetic field is first reduced adiabatically to a low level; e.g., 1 G. Then a sudden reversal to \(-1 \text{ G}\) takes place. If the reversal is so sudden that the atoms cannot follow it, the states become, with respect to the new magnetic field direction, different quantum states as follows:

<table>
<thead>
<tr>
<th>Hydrogen</th>
<th>Deuterium</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 \to 4</td>
<td>1 \to 4</td>
</tr>
<tr>
<td>2 \to 2</td>
<td>2 \to 3</td>
</tr>
<tr>
<td>3 \to 1</td>
<td>3 \to 2</td>
</tr>
<tr>
<td>4 \to 4</td>
<td>4 \to 1</td>
</tr>
<tr>
<td></td>
<td>5 \to 6</td>
</tr>
<tr>
<td></td>
<td>6 \to 5</td>
</tr>
</tbody>
</table>

Thus, for hydrogen atoms, if one starts at high positive fields with an equal mixture of states 1 and 2 (0% polarization), after a sudden zero crossing followed by an adiabatic increase to a high negative field, one obtains an equal mixture of states 2 and 3 (100% polarization). For deuterium atoms, if at high positive fields we have an equal mixture of states 1, 2, and 3 (\(P_3 = P_{33} = 0\)), a sudden zero crossing followed by an adiabatic increase to a large negative field leads to an equal mixture of the states 2, 3, and 4 (\(P_3 = -2/3, P_{33} = 0\)). This process has been applied with good results at two laboratories.

Our concern here is the extent to which one can achieve these diabatic transitions for practical beam sizes and magnetic field shapes. Our attention will be focussed mainly on hydrogen or deuterium atoms in their 1-states. [For hydrogen there is no difference between state 2 for a very small positive and a very small negative field, so it is not meaningful to inquire whether the atom "followed" the field direction or not. For deuterium atoms in states 2 and 3, such a question is meaningful, but since equal initial populations of 2 and 3 are involved in the applications we have in mind, symmetrical transitions (or the lack thereof) between them are of no consequence.]  

Consider again the uniformly falling field for which \(B_z = -cz = -cvt\) and \(B_y = \omega cr\). At \(z = 0\) the (minimum) field is \(B = \omega cr\). The field angular rotation rate is, again,

\[\omega = \frac{-(r/2v)}{(r/2v)^2 + t^2} \tag{32}\]

Note that this frequency is independent of \(c\), the rate of fall of the \(B_z\) field. However, the minimum field, for given \(r\), is proportional to \(c\). Thus, increasing the rate of fall of \(B_z\) has the net effect of raising the minimum field and thus the ability of the atom to follow the field reversal is improved. Larger \(r\) increases both the minimum
Fig. 9. Fraction of the initial state 1 making the desired transition when the field is linearly reversed at the indicated rate. The abscissa is the radius with respect to the (cylindrically symmetric) field axis. Slightly different results are obtained for hydrogen state 1 (H) and deuterium state 1 (D), as indicated. A velocity of 30 cm/µsec is assumed.

Field and decreases the angular rate with which the field reverses, and thus also improves the ability of the atom to follow the reversal. From this discussion it appears that 1) the field must reverse as slowly as possible (contrary to one's first impression) and 2) there exists a maximum beam diameter, for a given rate of fall for Bz, for which the scheme will be applicable. As pointed out by Sona, the presence of transverse (stray) field components will place a lower limit on the field parameter c.

Figure 9 shows the fraction of the initial state 1 making transitions to state 3 for hydrogen or to state 4 for deuterium atoms, as a function of beam radius for several rates of fall for Bz. There is no appreciable difference between the results for hydrogen and deuterium. A priori, one would expect a larger fraction of the deuterium atoms to make transitions since the relevant precession frequency (with respect to which the field must rotate rapidly) is only two-thirds as large for deuterium as it is for hydrogen. The deviation from expectations is probably because, at a given (low) magnetic field, the deuterium atom is less well-described as a simple particle (with spin 3/2) than is the hydrogen atom (with spin 1).

Fig. 10. Results of the calculations presented in Fig. 9 expressed in terms of P (for hydrogen) and P3 and P33 (for deuterium). The polarizations are expressed with respect to the final field direction. A velocity of 30 cm/µsec is assumed.

Figure 10 presents the information given in Fig. 9 in terms of polarization versus beam radius. For hydrogen atoms this presentation carries no new information, but for deuterium atoms, one wishes to know of the effect on both vector and tensor polarization. It would be equivalent, for our purposes, to specify the relative population of each of the final states other than the initial state.

Depolarization Effects Associated with the Addition of Two Electrons to a Polarized H+ or D+ Ion Beam

If a positive H or D ion has two electrons added to it in a gas or foil, there may be a time interval during which the system is a neutral atom. During this time some depolarization will take place if a) the time interval is long enough and b) the magnetic field in the stripper region is small or zero; i.e., unless x >> 1. Similar arguments apply to the case of the stripping of two electrons by a negative ion. This point will be further discussed later.

Let us assume that the magnetic field, if any, in the "adder" region is constant. We choose our z-axis to be parallel with the field. The nuclear polarization may be parallel to the z-axis or inclined at some angle with respect to it. (The solutions presented in Section 2 are sufficiently
general to handle any such orientation.)

First, we consider hydrogen ions with the nuclear spin parallel to the stripper field. If we assume that the first electron is captured into the ground state, the atom will be in either state 1 or in state 4 (equal probability), where we refer to strong field states regardless of the actual field strength. From the general solutions already given, atoms which are in state 1 initially will remain so; thus, for these atoms, we have

\[ \langle \sigma_z \rangle = 1. \]  

(33)

The time dependence of \( \langle \sigma_z \rangle \) for an atom initially in state 4 is obtained from the general solution (Eq. 23) with \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0 \) and \( \varepsilon_4 = 1 \):

\[ \langle \sigma_z \rangle = \frac{1}{1 + \lambda} \left( x^2 + \cos \sqrt{1 + x^2} \frac{\Delta W}{\hbar} t \right); \]  

(34)

i.e., for \( x = 0 \) depolarization occurs with the normal hyperfine frequency \( \Delta W/\hbar \) as the characteristic rate. If the time interval \( \tau \) between the pick-up of the first and of the second electron is random, and long compared to \( \hbar/\Delta W \), then, for a beam of particles, \( \langle \sigma_z \rangle \) will have the average value \( x^2/(x^2 + 1) \). Combining Eqs. 33 and 34 we obtain for the overall beam polarization

\[ P = \frac{1}{2} \left[ 1 + x^2/(x^2 + 1) \right]; \]  

(35)

If the time \( \tau \) is sufficiently short, no depolarization occurs. For zero magnetic field \( (x = 0) \), we see that a maximum overall depolarization of 50% may occur; for large fields \( (x \gg 1) \), no depolarization will occur. Consideration of the case where the initial nuclear polarization and stripper field are antiparallel gives, except for overall sign, a result identical to that given above. That is, for an atom initially in state 3

\[ \langle \sigma_z \rangle = -1, \]  

(36)

while for an atom initially in state 2

\[ \langle \sigma_z \rangle = -\frac{1}{1 + \lambda} \left( x^2 + \cos \sqrt{1 + x^2} \frac{\Delta W}{\hbar} t \right). \]  

(37)

Thus, the overall polarization for an equal mixture of states 2 and 3 may be written

\[ P = \frac{1}{2} \left[ 1 + x^2/(x^2 + 1) \right]; \]  

(38)

In summary, if depolarization is to be avoided, either the time between the first and second collisions must be small compared to \( \hbar/\Delta W \) or a large magnetic field must be present.

If the nuclear spin is perpendicular to the adder magnetic field direction (z-axis), the (equally probably) initial states are as follows: for the electron spin parallel to z, \( \langle \sigma_z \rangle = 1 \) implies \( \varepsilon_1 = \varepsilon_2 = 1/\sqrt{2} \) and \( \varepsilon_3 = \varepsilon_4 = 0 \); if the electron spin is antiparallel to z, \( \langle \sigma_z \rangle = 1 \) implies \( \varepsilon_1 = \varepsilon_2 = 0 \) and \( \varepsilon_3 = \varepsilon_4 = 1/\sqrt{2} \). For the first of these initial conditions we obtain

\[ \langle \sigma_z \rangle = \frac{1}{2} (1 - x/\sqrt{2}) \cos \left[ \left( 1 + x/\sqrt{2} \right) \frac{\Delta W}{\hbar} t \right] + \frac{1}{2} (1 + x/\sqrt{2}); \]  

\[ x \ll \Delta W/\hbar \]  

(39)

where \( \beta = (1 + x^2)^{1/2} \). For the second initial condition we obtain

\[ \langle \sigma_z \rangle = \frac{1}{2} (1 + x/\sqrt{2}) \cos \left[ \left( 1 - x/\sqrt{2} \right) \frac{\Delta W}{\hbar} t \right] + \frac{1}{2} (1 + x/\sqrt{2}); \]  

\[ x \gg \Delta W/\hbar \]  

(40)

For very large \( x \), Eqs. 39 and 40 each become

\[ \langle \sigma_z \rangle = \cos \left( \frac{\Delta W}{\hbar} t \right); \]  

\[ x \ll \Delta W/\hbar \]  

(41)

i.e., the polarization precesses at the expected high field rate. For small fields Eqs. 39 and 40 each become

\[ \langle \sigma_z \rangle = \frac{1}{2} (1 + \cos \left( \frac{\Delta W}{\hbar} t \right)); \]  

\[ x \gg \Delta W/\hbar \]  

(42)

i.e., again depolarization occurs with the normal hyperfine frequency as the characteristic rate. (The zero field limit must, of course, be independent of the direction of the assumed angle between the nuclear polarization and the magnetic field.) A more general orientation of the magnetic field axis may be considered with the aid of the solutions given in Section 2.

For deuterons the situation is similar but slightly more complicated. The vector and tensor polarization \( P_3 \) and \( P_{33} \) for each of the (strong field) initial states 1-6 is as follows:
where \( \delta t = (x^2 + x + 9/4)^{1/2} \). We note that, at zero field, the time dependence becomes of the form \( \cos(\Delta W/\hbar) \), so once again depolarization occurs at the normal hyperfine frequency.

For a deuteron beam initially in the \( m_z = 1 \) state with respect to the adder field direction, strong field states 1 and 6 are populated with equal probability; the time-averaged polarization parameters become

\[
P_3 = \frac{1}{\beta^+} \left( 1 + \frac{2x^2 - x + 9/4}{x^2 - x + 9/4} \right) \quad (\hbar/\Delta W \ll \tau ) ,
\]

\[
P_{33} = \frac{1}{\beta^-} \left( 1 - \frac{2x^2 - x + 9/4}{x^2 - x + 9/4} \right) \quad (\hbar/\Delta W \ll \tau ) ,
\]

Similarly, for \( m_z = 0 \) we obtain

\[
P_3 = \frac{1}{\beta^+} \left[ \frac{1}{x^2 + x + 9/4} - \frac{1}{x^2 - x + 9/4} \right] \quad (\hbar/\Delta W \ll \tau ) ,
\]

\[
P_{33} = \frac{-1}{\beta^-} \left[ \frac{1}{x^2 + x + 9/4} + \frac{1}{x^2 - x + 9/4} \right] \quad (\hbar/\Delta W \ll \tau ) ,
\]

and, for \( m_z = -1 \)

\[
P_3 = -\frac{1}{\beta^+} \left( 1 + \frac{2x^2 - x + 9/4}{x^2 - x + 9/4} \right) \quad (\hbar/\Delta W \ll \tau ) ,
\]

\[
P_{33} = \frac{1}{\beta^-} \left( 1 + \frac{2x^2 - x + 9/4}{x^2 - x + 9/4} \right) \quad (\hbar/\Delta W \ll \tau ) .
\]

For \( \hbar/\Delta W \gg \tau \) no depolarization occurs. For zero field Eqs. 44-46 reduce to

<table>
<thead>
<tr>
<th>Initial State</th>
<th>( \frac{P_3}{\beta^+} )</th>
<th>( \frac{P_{33}}{\beta^-} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7/9</td>
<td>1/3</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-2/3</td>
</tr>
<tr>
<td>-1</td>
<td>-7/9</td>
<td>1/3</td>
</tr>
</tbody>
</table>

That is, the initial vector polarization is reduced to seven-ninths of its initial value while the initial tensor polarization is reduced to one-third of its initial value. We may summarize the zero field results as follows:

<table>
<thead>
<tr>
<th>Characteristic Frequency for Depolarization</th>
<th>Maximum Depolarization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Protons 1420 MHz</td>
<td>1/2</td>
</tr>
<tr>
<td>Deuterons 327 MHz</td>
<td>2/9</td>
</tr>
<tr>
<td>Tritons 1517 MHz</td>
<td>1/2</td>
</tr>
</tbody>
</table>

In the above it is assumed that the intermediate atoms are formed in the ground state. At higher fields the depolarization is smaller; however, note that the characteristic frequency for depolarization becomes larger.

In the above discussion, it has been assumed that the first electron is added in the \( 1S_{1/2} \) state. However, the theory holds for capture into any \( J = 1/2 \) state, so long as the atom remains in this state until the second electron is added and provided that the appropriate hyperfine splitting is used in the description. If radiative decay occurs, some additional depolarization will result. On the other hand, relatively small magnetic fields will be sufficient to produce a strong field with respect to the higher hydrogen-atom excited states, and radiative decay in the presence of a strong field will result in no nuclear depolarization. Thus, a field strength which is sufficient to prevent depolarization of a \( 1S_{1/2} \) state is also sufficient to prevent depolarization while an atom is in an excited state.
and during the decay of the atom (eventually to the \(1s_{1/2}\) state).

For a "thick" adder, as required for a large negative ion yield, electrons may be added and subtracted several times before the particle escapes as a negative ion. The considerations above will apply to each time interval during which the particle exists as a neutral atom. For example, consider a proton which is converted to an \(H^-\) ion via the process \(H^+ + H^0 + H^+ + H^0 + H^-\). If we assume zero magnetic field and that both of the time intervals during which the particle is neutral are large compared to \(\hbar/\Delta W\), a depolarization of 75\% (i.e. \(1 - (\frac{1}{2})^2\)) would be expected. For deuterons the vector depolarization would be \(1 - (\frac{7}{2})^2\) and the tensor depolarization would be \(1 - (\frac{3}{2})^2\). In practice, of course, both the number of charge exchange events and the interval between them will be random.

The stripping of two electrons from a negative ion presents a somewhat similar problem. However, a hydrogen negative ion has a diffuse wave function compared to a hydrogen atom. Thus, if one of the electrons is suddenly removed, the remaining electron will tend to be spread over a relatively large region. Stated differently, the atom will have a high probability of being produced in a state other than the ground state. Therefore, the effects discussed in the previous paragraph are expected to be more important in the stripping case than in the adding case. It is possible that studies of depolarization versus magnetic field could yield information about 1) the wave function of the hydrogen negative ion and 2) the nature of the collisions which induce radiative decay of the higher hydrogen atom states that are produced in the partial stripping of an \(H^-\) ion.

ACKNOWLEDGMENTS

It is a pleasure to thank Donald Dodder for a number of clarifying discussions. I also appreciate conversations with Bailey Donnally, Ralph Stevens, Joe McKibben, John Gammel, and Charles Drake.

REFERENCES

A COMPUTER CODE FOR SOLVING THE ONE-ELECTRON SCHROEDINGER EQUATION

This code solves the time-dependent Schrödinger equation for a Hamiltonian of the form

$$\hat{H} = \hat{H}_0 + \left( \mu_0 g_J \hat{J} + v_N \hat{S}_I \cdot \hat{I} \right) \cdot B + a(\hat{I} \cdot \hat{J})$$

where the notation is defined in the preceding pages. An arbitrary time dependence of the external magnetic field B is allowed. A numerical integration of the resulting set of linear first-order differential equations is carried out with automatic error control.

The input for the code is as follows:

Card 1 FORMAT (16)

NZ number of points at which axial and radial field table is to be specified

= 0 if no table to be specified

Card 2 to Card (NZ + 1) FORMAT (3F12.6)

ZZ(I) axial position (cm)

BZZ(I) axial field strength (G)

BRR(I) radial field strength 1 cm off axis (G)

CARD NZ + 2 FORMAT (316)

M type of atom

= 1: for hydrogen atoms

= 2: for deuterium atoms

= 3: for tritium atoms

NSTATE initial state

= 1 to 4 if hydrogen or tritium atoms intermediate field states

= 1 to 6 if deuterium atom intermediate field states 1 to 6

= 11 to 14 if hydrogen or tritium strong field states 1 to 4

= 11 to 16 if deuterium atom strong field states 1 to 6

MODE option for defining magnetic field

Card NZ + 3 FORMAT (3F12.6)

TI initial time (usec)

TTF final time (usec)

HHP time interval between output steps (usec)

Card NZ + 4 FORMAT (6F12.6)

BZ1 initial axial field (G)

BZ2 final axial field (G)

BRMAX final transverse field (G)

Y1 distance off axis (cm)

PHI azimuthal angle transverse field makes with respect to x axis (deg)

VELOC particle beam velocity (cm/usec)

Card NZ + 5 FORMAT (316)

Z1 position at which axial field begins to decline or to be defined by table (cm)

Z2 position at which axial field assumes a constant final value (cm)

Z3 position at which transverse component begins to rise as sine-squared function (cm)

Z4 position at which transverse component assumes a constant final value (cm)

The code renormalizes the state vector to unity total probability before each print to remove accumulated normalization errors (via subroutine RENORM). If this error exceeds 1%, an error message is printed.

The time, field components and total field, nuclear polarization, electron polarization, and
squared amplitudes of the four (or six) strong field states are printed with reference to the coordinate system as defined by the user. These results are then printed a second time in terms of the intermediate field states with respect to a z axis defined by the instantaneous direction of the total field. In the second set of output the field is specified in terms of azimuthal and polar angles, and beam displacement is given instead of elapsed time.

The program consists of a main program MRANNA together with a number of subroutines. The function of each of the subroutines is briefly described below.

**SUBROUTINE SETUP**

This subroutine reads in parameters necessary to specify the B field as described above. Certain often-used combinations of the input parameters are computed here.

**SUBROUTINE BFIELD (T, BX, BY, BZ)**

This subroutine computes the field components BX, BY, BZ at the time T, assuming Z = VELOC*T, X = 0, and Y = Y1.

**SUBROUTINE DERIV (T, V, FD)**

This subroutine computes the values of the first derivatives FD(I) (I = 1 to 8 or I = 1 to 12) given the value of T (time) and of the variables V(I) (I = 1 to 8 or I = 1 to 12). [Note that four (six) complex first-order differential equations result in eight (twelve) real first-order differential equations.]

**SUBROUTINE PRINT (T, VS)**

This subroutine prints the first type of output described above at the specified times and also stores the second type of output for later printing.

**SUBROUTINE ROT (M, ALPHA, BETA, GAMMA, V)**

This subroutine rotates the state vector V through the Euler angles α, β, and γ.

**SUBROUTINE REMOR (N, V, VV)**

This subroutine renormalizes the state vector V so that it has unity total length.

**FUNCTION ARCTAN (Y, X)**

This function computes arctangents, in degrees, for all zero and nonzero values of Y and X.

**SUBROUTINE TABLE (Z, ZZ, NZ, I, IFLAG)**

This subroutine performs a table look-up in the ordered table ZZ.

**SUBROUTINE INTEG (MN, TI, TTF, HH, HNP, NM, VVM, IP, XO, TT, XXP)**

This subroutine integrates an arbitrary system of real linear differential equations. The arguments of this subroutine are defined by comments in the main program listing. The monitoring feature (a periodic test of a specified variable against some limit) is not used. INTEG together with the subroutines START, RNGA, ACCRY, TEST, DIODE, ADAMS, and DOUBLE constitute the complete integration package.

```
PROGRAM MRANNA (INPUT=OUTPUT=FILM, TAPE=12=FILM)
C UNITS MICROSCONS
COMMON/BLCK/141,A2,A3,A4,C1,RO
COMMON/BLCK/PR50(6X)201,TIM0(201),8REX(201),8RV(201),8RBZ(201),
1P3(201),PR73(201),REL(201),N+TIME
COMMON/BLCK1/M,N
COMMON/BLCF/T1,T2,T3,T4,T5,T6,BZ1,BZ2,BRMAX,CPHI,SPHI,PI,P,Y,MODF
COMMON/BFL/DP/ZZ(100),BZZ(100),BRR(100),NZ
UTLEMUION X0(30),XXP(30),BPL0T(201)
COMPLEX C
1 FORMAT(10I6)
2 FORMAT(1H1)
3 FORMAT(* PROTONS 6F14.8)
4 FORMAT(* NEUTrons 6F14.8)
5 FORMAT(* TRITONS 6F14.8)
6 FORMAT(12F6.3)
7 FORMAT(* TIME RX BY BZ 8 P1
2 4)
8 FORMAT(14F9.4)
```
9 FORMAT(6F12.4)
10 FORMAT(* STRONG FIELD STATE COEFFICIENTS, INITIAL COORDINATE SYSTEM)
11 FORMAT(* INTERMEDIATE FIELD STATE COEFFICIENTS, COORDINATE SYSTEM)
12 FORMAT(* STRONG FIELD PARAMS, INITIAL STRONG FIELD COEFFS)
13 FORMAT(* STRONG FIELD PARAMS, INITIAL STRONG FIELD COEFFS)
14 FORMAT(* STRONG FIELD PARAMS, INITIAL STRONG FIELD COEFFS)
15 FORMAT(* STRONG FIELD PARAMS, INITIAL STRONG FIELD COEFFS)

C Mode, or 3 for protons, deuterons, tritons
C NSTATE = 1-4 or 1-6 for intermediate field initial states
C NSTATE = 11-14 or 11-16 for strong field states
C MODE = 1 or 2 on BFIELD OPTION (SEE COMMENTS IN RFIFLD)

99 X(1)=0.0
IF(M=2)101,102,103
101 H=6.3*448/1.0
A1=0.5/(1.0+0.015221.0+1.522)
A2=0.5/(1.0+0.015221.0+1.522)
A4=0.0
X=MTOT/6.3448
IF(NSTATE,GE,11) GO TO 144
143 NUS=SQRT(0.5*0.5*0.5*0.5)/SQRT(1.0*X**2)
144 H=(1.0+1.52+1.52+1.52) HSTATE
150 X(1)=1.
Gn TO 104
151 X(3)=NUS
X(7)=NUS
Gn TO 104
152 X(5)=1.
Gn TO 104
153 X(3)=D+NUS
X(7)=D+US
Gn TO 104
154 NSTATE=IFESTATE-10
155 X(2)*NSTATE-1=1.
Gn TO 104
102 H=14.05/1.0
A0=0.5/(1.0+0.000233)
A1=0.5/(1.0+0.000233)
A2=0.5/(1.0+0.000233)
A3=0.0673/1.0
A4=0.0
X=4*A1+A2+A3+A4+A0
X=MTOT/14.05
IF(NSTATE,GE,11) GO TO 144
143 NUS=X*1.0/3.0/SQRT(1.0*2.0*X**3.0*X**2)
D+NUS=(X*1.0/3.0)/SQRT(1.0*2.0*X**3.0*X**2)
GO TO (140,141,142,143,144,145)*NSTATE
140 Xn(1)=1.0
GO TO 104
141 Xn(3)=SQRT(0.5+0.5*DPLUS)
Xn(11)=SQRT(0.5+0.5*DPLUS)
GO TO 104
142 Xn(5)=SQRT(0.5+0.5*DMINUS)
Xn(9)=SQRT(0.5+0.5*DMINUS)
GO TO 104
143 Xn(7)=1.0
GO TO 104
144 Xn(5)=SQRT(0.5+0.5*DPLUS)
Xn(11)=SQRT(0.5+0.5*DPLUS)
GO TO 104
145 Xn(3)=SQRT(0.5+0.5*DMINUS)
Xn(9)=SQRT(0.5+0.5*DMINUS)
GO TO 104
146 NSTATE=NSTATE-10
Xn(2*NSTATE-1)=1.0
GO TO 104
103 Nn=67/55/1.0+16/23
A1=0.5*(1.0+0.001623/1.001623)
A2=0.5*(1.0+0.001623/1.001623)
A3=0.5*(1.0+0.001623/1.001623)
A4=0.5+0.0
Nn=4
POINT 5,A1,A2,A3,A4,A0
TB=TNT/67.7555
Go To 99
104 A=0.5*(2*N+1.165927+1.401+1.6413)
C=CHPLX(7.0,-36.0)
C REFER INITIAL VECTOR TO PROBLEM DEFINED Z AXIS VIA APPROPRIATE ROTATION
CALL ROTN(1,ALPHA,RETA,GAMMA,XN)
Nn=2*N
POINT 10
POINT 7
Call PRINT(TI,XO)
HH=0.0001
MM=0
VIV=0.0
ABS=1.0E-5
REL=1.0E-5
Call PRINT(TI,XO)
MM=M“They produced a plot with LASL subroutines
C6LL ADV(I)
CALL DGA(500,50,50,800,TT,TF,HH,MM,ABS,REL,XO,TT,XP)
C6LL SLLIN (10,50,20,70,0,0)
C6LL SRLIN (10,50,20,70,0,0)
C6LL SRLIN (10,50,20,70,0,0)
C6LL SRLIN (10,50,20,70,0,0)
10 CONTINUE
C THE FOLLOWING 18 CURVES PRODUCE A PLOT WITH LASL SUBROUTINES
CALL ADV(I)
CALL DGA(500,50,50,800,TT,TF,HH,MM,ABS,REL,XO,TT,XP)
CALL DLNL2(10,50,20,70,0,0)
CALL SLLIN (10,50,20,70,0,0)
CALL SRLIN (10,50,20,70,0,0)
On 120 J=IoN
On 121 I=IONSME
121 HPLOT(I)=QSQ(JCI)
120 CALL PLOTINTIMEoTTME (1)ojc13PLoT (1)01s4401)
CALL DGA(so9q5n050~500~TT*TTFol .09-2.n)
CALL SRL1N(6,01)
CALL PLOT (NTIMEoTTME (1)1IO1OP33 (1)01,1600)
CALL PLOTfNTIMEtTIME (l)o10P3 (1)?1,63,0)
CALL LINCNT[60)
PHI=PHI*1ROoO/3.1415927
WRITE(~2~12) 71,Z2,Z3,Z4S75sz6,BZ1 *BZ2,RRMAX$Yj,PHI OVELOC
WPITE(12013) MONSTATE0
{XO(I)QI=l$NN)
CALL ADV(I)
PRINT 13*!f,Ns7ATE. (XOIIl .1=1 sNNl
Go To 100
END)

SUBROUTINE SETUPA
C READ= IN FIELD PARAMETERS AND COMPUTES SOME FREQUENCYLY USED QUAN-
COMM/FI D/TloT~oT3,T4,T5,T6,BZl ,Bz2,BRMAX,CDb11,SPH1,P1P$Y2.MOOF,SETR0013
C DISTANCES IN CM FoR FIELD PARAMETER ENTRY
2 FORMAT(~F12.s)
PT23.3,1415927/2.e0
RFAD 2,8.1=BZ2|BRMAX,YP11=MI+VFLOC
P0INT2+Z1+Z2,BZ2,BRMAX,Y1=OMI+VFLOC
RFAD 2,Z1,Z2,Z3*74*Z5*26
P0INT2+Z1,Z2,Z3*74*Z5*26
Ti=Z1/VELOC
T2=Z2/VELOC
T3=Z3/VELOC
T4=Z4/VELOC
T5=Z5/VELOC
T6=Z6/VELOC
C Y1 IN CM, PNI IN DEGREES
CPH1=COS(PHI*PI2/90.e0)
SOMI=SIN(PHI*PI2/90.e0)
Go To(100,200*,3000),MODE
100 Y2=Y1
RETURN
200 Y2=-(Y1/2.0)*PI2*(BZ1-RZ2)/VELOC*(T2-T1))
RETURN
300 Y2=(RZ1-RZ2)*Y1/(2*PI2*VELOC*(T2-T1))
RETURN
C MODE 1 (READ IN TABLES OF BZ,RR)
C COMPUTES FIELD COMPONENTS AT POSITION Z=VELOC*T, Y=Y1, X=O
CALL TABLF (70ZZ(1) SNZ91,MFLAG)
0nzDz=(8ZT (Iol)-f17z(I) )/f7Z(I*l)-Z7(I))
DuRDZ9(RHR(X*1 )-RRR(I) )/(ZZ(I*l)-ZZ(X))
Y7NORN=BZ7(I)
BuNORM=RRP(I)
Z=VELOC*T
RETURN
C TABLE IN GAUS ON AXIS AND 1 CM OFF AXIS NORMALIZED TO UNITY BZ(1)
CALL TABLE (7*ZZ(1) )NZ,1,MFLAG)
ON-ZDZRZ7(1)*RZ7(1)/7Z1(I)*Z7(1))
ON-N07#BAP11)=R07(1)/7Z1(I)*Z7(1))
Y=NORM=BZ7(1)*OB2NZ*(Z-Z7(1))
BUNORM#R07(1)*OB2NZ*(Z-Z7(1))
21
H7=H7C*H71*H72*ZC*Z2

C INITIAL RAY DEFINES Y-Z PLANE
BY=Y2*(R2)*H72*H71*ZC*Z2
H7=0.0
Gn To 500
101 H7=B72
H7=0.0
H7=0.0
Gn To 500

C MODE 2 (COSINE-SQUARED FIELD DECAY)
300 IF (T>T0) Gn To 301
H7=0.0
BY=0.0
H7=B71
Gn To 500
301 IF (T>T0) Gn To 302
ANG=I2*(T-T1)/(T2-T1)
H7=B72*(H71-H72)*COS(ARG)*Z
H7=Y2*SIN(2.*Z*ANG)
H7=0.0
Gn To 500
302 H7=0.0
BY=0.0
H7=B72
Gn To 500

C MODE 3 (EXPONENTIAL FIELD DECAY)
400 IF (T>T0) Gn To 401
H7=0.0
BY=0.0
H7=B71
Gn To 500
401 EANG=EXP(-((T-T1)/(T2-T1))
H7=(H71-H72)*EARG+B72
H7=0.0
H7=Y2*EARG
Gn To 500

C START TRANSVERSE COMPONENT, PHI IS ANGLE WITH REFLECT TO X AXIS
C INCIDENT HEAT IN Y-Z PLANE
500 IF (BRMAX+FQ>0.0) RETURN
IF (T>T0) Gn To 501
H7=0.0
Gn To 505
501 IF (T>T0) Gn To 502
H7=BRMAX*Sin(P12*(T-T3)/(T4-T3))**2
Gn To 505
502 H7=BRMAX
405 H7=BRMAX*SMT*AY
BY=BR*CPMT+H7
RETURN
END

SUBROUTINE DERIV(T,VT,FD)
C COMPUTES THE FIRST DERIVATIVE OF THE STATE VECTOR FOR USE BY INTEG
UTMENSION V(T),F(DT)
COMMON/HLCK1/AL,A2,A3,A4,C1,R0
COMMON/HLCK7/HNN
COMMON/HLCK3/H13
COMMON/HLCK4/H14
COMMON/HLCK5/H15
COMMON/HLCK6/H16
CALL RFT6L(D(T,T),(T))
END

C SPIN 1/2 SECTION
C A1=0.5*(1-K) A2=0.5*(1-K) A3=0.5*K A4=0.0
C B0=DPL&TW/(M**G)
H1=CMLPLX(V(1)+V(2))
B2=CMLPLX(V(3)+V(4))
H1=CMLPLX(V(5)+V(6))
A4=CMLPLX(V(7)+V(8))
A4=0.0
B0=CMLPLX(C1+M10J)/(H8AB*1) UNITS 1/MICROSECOND
F1(1)=REAL(B0)
F1(2)=AIMAG(B0)
F1(3)=REAL(B1)
F1(4)=AIMAG(B1)
F1(5)=REAL(B2)
F1(6)=AIMAG(B2)
F1(7)=REAL(B3)
F1(8)=AIMAG(B3)
F1(9)=REAL(B4)
F1(10)=AIMAG(B4)
F1(11)=REAL(B5)
F1(12)=AIMAG(B5)
F1(13)=REAL(B6)
F1(14)=AIMAG(B6)
F1(15)=REAL(B7)
F1(16)=AIMAG(B7)
F1(17)=REAL(B8)
F1(18)=AIMAG(B8)
F1(19)=REAL(B9)
F1(20)=AIMAG(B9)
F1(21)=REAL(B10)
F1(22)=AIMAG(B10)
F1(23)=REAL(B11)
F1(24)=AIMAG(B11)
F1(25)=REAL(B12)
F1(26)=AIMAG(B12)
F1(27)=REAL(B13)
F1(28)=AIMAG(B13)
F1(29)=REAL(B14)
F1(30)=AIMAG(B14)
HnOT=Cl*[(A3*MINUS*B1+(A2*Z=0.25*RO)*R2+0.5*PLUS*B3+0.5*RO*RO)+RDLIS*B3+0.5*RO*R4] 0ERIO031
Fn(3)=REAL(ROOT) nERIO031
Fn(4)=AIMAG(ROOT) nERIO032
HnOT=Cl*(n15*MINUS*B2*(n15*RO=11B)113+A3*MINUS*B4) nERIO033
Fn(5)=REAL(ROOT) nERIO034
Fn(6)=AIMAG(ROOT) nERIO035
HnOT=Cl*(n15*MINUS*B1+0.5*RO*RO+A3*PLUS*B3=(A2*Z=0.25*RO)*R4) nERIO036
Fn(7)=REAL(ROOT) nERIO037
Fn(8)=AIMAG(ROOT) nERIO038
RETURN
100 CONTINUE
C SPIN 1 SECTION
A1=0.5+K A2=0.5-K A3=K A4=SQRT(0.5)
C NO=(L1+FLAT)/(105*MU.GJ) clarM11*(3J/(APB1*T) UNITS l/MICROSD
A1=CMPLX(V(1)) OV(2)) nERIO042
A2=CMPLX(V(3) 9V(4) ) nERIO043
A3=CMPLX(V(5) 9V(6) ) nERIO044
A4=CMPLX(V(7) 9V(8) ) nERIO045
A5=CMPLX(V(9) 9V(14) ) nERIO046
A6=CMPLX(V(11) 9V(12) ) nERIO047
HnOT=Cl*[(A1*H7*0.5*BO)*R1+A3*PLUS*R2+0.5*PLUS*B3 115B6) nERIO050
Fn(1)=REAL(ROOT) nERIO051
Fn(2)=AIMAG(ROOT) nERIO052
HnOT=Cl*(A3*MINUS*B1+0.5*RO*RO+A3*PLUS*B3+0.5*PLUS*B5 144*RO*RO4) nERIO053
Fn(3)=REAL(ROOT) nERIO054
Fn(4)=AIMAG(ROOT) nERIO055
HnOT=Cl*(A3*MINUS*B2*(A2*Z=0.5*BO)-R3*0.5*PLUS*B4 144*RO*RO4) nERIO056
Fn(5)=REAL(ROOT) nERIO057
Fn(6)=AIMAG(ROOT) nERIO058
HnOT=Cl*(n15*MINUS*B3*(n15*RO=A1*RO)114+A3*MINUS*B5) nERIO059
Fn(7)=REAL(ROOT) nERIO060
Fn(8)=AIMAG(ROOT) nERIO061
HnOT=Cl*(n15*MINUS*B2+A4*RO*RO3+0.5*PLUS*B3-0.5*PLUS*B4) nERIO062
HnOT=Cl*[(A1*H7*0.5*BO)*R1+A3*PLUS*R2+0.5*PLUS*B3 115B6) nERIO063
Fn(1)=REAL(ROOT) nERIO064
Fn(2)=AIMAG(ROOT) nERIO065
HnOT=Cl*(A3*MINUS*B1+0.5*RO*RO+A3*PLUS*B3+0.5*PLUS*B5 144*RO*RO4) nERIO066
Fn(3)=REAL(ROOT) nERIO067
Fn(4)=AIMAG(ROOT) nERIO068
RETURN
END

SUBROUTINE PRINT1(T,V,S)
C PRINTS STRONG FIELD SQUARED AMPITUDINES WITH RESPECT TO PROBLEM DF-
C FINE) Z AXIS. STORES WEAK FIELD SQUARED AMPITUDINES WITH RESPECT
C TO INSTANTANEOUS A AXIS. STORES WEAK FIELD SQUARED AMPITUDINES WITH RESPECT
C COMMON/LOCK-20011,TIME(201) 11AX(201) BAY(201) 11AZ(201)
C I(TIME(I))=PIT(201) 11EL(201) IN TIME
C COMMON/LOCK-20011
C DIMENSION V(12) 11V(12) 11V(12)
C TIME=TIME+1
DN 90 TIME=1
40 V(I)=VS(I)
CALL MENUV(IN,VS)
LC(IV,IB,+1) PRINT A0
AN FORMAT(* RENORMALIZATION EXCEEDS 1 PERCENT*)
CALL BR-PRINT(1+AX+AY+AZ)
HAX(1)TIME)+AX
HAY(TIME)+AY
HAX(TIME)+HAY
HTOT=SQT(A**2+B**2+C**2)
A PHAE=(1.415297/180.0)*ARCTAN(BY/AX)
HRT=(3.145927/180.0)*ARCTAN(SQRT(A**2+BY**2)) 11RZ
GAMMA=0.0
GO TO 101
100 HNMT=HAX/H3
HMT1=SQT(0.5+0.5X/SQRT(1+0.5X**2)) I1HIS=SQRT(0.5+0.5X/SQRT(1+0.5X**2)
DN 105 T=I
105 HASQ(I)=V(I)*V(I)**2+V(2*I)**2
PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)

PPEL=B5S(1)+ARSO(2)+BBSn(3)=ARSO(4)
PPEL=BS(1)+ARSO(2)+BBSn(3)=ARSQ(4)
12 \text{ Return}
\begin{align*}
\sin (\theta) &= \frac{\text{opposite}}{\text{hypotenuse}} \\
\cos (\theta) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\
\tan (\theta) &= \frac{\text{opposite}}{\text{adjacent}}
\end{align*}
\begin{align*}
\sin (\theta) &= \sin (\theta + 2\pi) \\
\cos (\theta) &= \cos (\theta + 2\pi) \\
\tan (\theta) &= \tan (\theta + \pi)
\end{align*}
SUBROUTINE TABLE (Z+ZZ*N7+I*MFLAG)
C WRITERS INDEX I OF NEXT SMALLER ENTRY OF ZZ(1)
C MFLAG=1 IF Z=ZZ(I) FOR SOME I
DIMENSION ZZ(100)
N=1
MFLAG=1
IF (Z.LT.Z7(I)) GO TO 30
IF (Z.GT.Z7(N)) Go To 30
DO 10 T=1+M7*10
IF (Z.EQ.77(I)) MFLAG=0
I=Z.LT.Z7(I)) GO TO 20
10 CONTINUE
N1=N2=1
GO TO 11
11 On 21 J=1,MJ
IF (Z.EQ.77(I+J)) MFLAG=0
IF (Z.LT.Z7(I+J)) GO TO 40
21 CONTINUE
40 I=I+J-1
RETURN
30 FORMAT (* OUT OF RANGE OF TABE*)
1=1
RETURN
END

SUBROUTINE INTE(NN,TF,TF,TH,TH,HH,HH,PR,MM,VM,VM,ABS,REL,XX,TT,XX)
C INTE SOLVES A SYSTEM OF NN FIRST ORDER DIFFERENTIAL EQUATIONS BY
C A 4TH ORDER ADAMS PREDICTOR-CORRECTOR METHOD WITH AUTOMATIC ERROR
C CONTROL. STARTING IS BY THE RUNGE-KUTTA METHOD
LOGICAL ACC
COMMON/INT/N,T,TF,H,H0,HP,H,H,M,M,ACC,XLB,RELTST,ABSTST,FAC70R,BN
DIMENSION XQ(30),XP(30),G(30,4),TD0UBL,NN,URL
C SET UP INITIAL VALUES
N=NN
TF=TF
TH=TH
HP=HP
HM=HM
VM=VM
On 10 I=1,N
10 X(I)=XQ(I)
T=TI
H0=TI+HP
H=H
AMSTST=ABE
RELTST=REI
FACT0R=HE; TST/ABSTST
X1=0.0066*RELTST
INDOUBL=0
NN,URL=3
M=2.*NH
30 CALL START(I*RET11)
G0 TO (1G0,99,1*IRET)
C SHOULD ANY OF THE STARTING VALUES BE PRINTED OUT
100 T=3.04H
On 3J M=2.4
T=TH
CALL TEST(I*RET11)
G0 TO (35,60,1*IRET)
35 CONTINUE

30H ACTAN=ACCTAN(180.0/3.1415927)
RETURN
END
C BEGIN ADAMS METHOD
40 CALL ADAMS
    CALL ACCRY
    IF (ACC) GO TO 50
    ON 45 I=1,N
45 X(I+1)=X(I)+1
    Go TO 30
50 CALL TEST(I,IRET)
    GO TO (101:100,IRET)
101 CALL OMPH(I,IRET)
    GO TO (40,401:400,IRET)
60 IF (J+5=9) GO TO 45
    ON 66 I=1,N
66 XP(I)=X(I,J)
65 CALL PRINT(T,XP)
    T=T
    ON 70 I=1,N
70 X(I)=XP(I)
90 RETURN
END
SUBROUTINE START (IRET)
C RUNGAKUTTA STARTING METHOD
    LOGICAL ACC
    COMMON/INT/N,T,TF,H,HP,H1,H2,LYVM,J,ACC,XYB,RETST,ABSTST,FACTOR,BND
1) J=2
    CALL RNGA
10 ON 15 I=1,N
15 X(I)=X(I)/2
20 IF ((T+H).NE.T) GO TO 30
    POINT 20
20 FORMAT(5H EQU NS CANNOT BE SOLVED FURTHER WITHIN GIVEN ERROR )
    T=TRSH=T+H
    POINT 21:TRUSH=T
21 FORMAT(AH T=T= T15,10, AH T= E15,10 )
    I=IRET
    RETURN
30 ON 40 J=M,3
40 CALL RNGA
41 CALL ACCRY
    IF (NOT ACC) GO TO 10
    J=4
    CALL RNGA
    I=IRET
    RETURN
END
SUBROUTINE RNGA
C INTERPARE AT POINT AHEAD ON THE J/TTH STEP OF LENGTH H.
    COMMON/INT/N,T,TF,H,HP,H1,H2,LYVM,J,ACC,XYB,RETST,ABSTST,FACTOR,BND
1) G(I)=H*F(I,J-1)
10 X(I,J)=X(I,J-1)+0.5*G(I,J)
    T=T+0.5*H
    CALL DERIV(TXX(I,J),F(I,J))
    ON 20 I=1,N
20 G(I)=H*F(I,J)
20 X(I,J)=X(I,J-1)+0.5*G(I,J)
    CALL DERIV(TXX(I,J),F(I,J))
    ON 30 I=1,N
30 G(I)=H*F(I,J)
    ON 40 I=1,N
CALL DERIV(T*X(I,J)+F(I,J))
ON 40 I=1,N
G(I,J)=RNGAO07
40 X(I,J)=X(I,J-1)+(G(I,J)+G(I,J)*G(T04))/60
RETURN
END

SUBLROUTINE ACCRY
C TESTS ANH REL ERROR AND SETS ACC .FALSE. IF EITHER SATISFIED

LOGICAL ACC
COMMON/INT/N,T,F,H,HO,H0,VM,VJ,ACC,XLB*RELST*ABSTST*FACTNO.ANN*ACCYOO1
1X(30,5),F(30,5),E(30),XP(30),G(30,4),TDOUHL,NNURL
ACC=.TRUE.
ON 50 I=1,N
E(I)=ABS(YP(I)-X(I,J))
IF (E(I) .GE.ABS (X(I,J)) AND (X(I,J)<1E-7)) GO TO 10
E(I)=E(I)*FACT0W
GO TO 50
10 IF (E(I) .GE.ABSTST) GO TO 20
E(I)=E(I)*FACT0W
GO TO 90
20 T=T-H
H=0.5*H
ACC=.FALSE.
75 FORMAT(1H , 16HSTST SIZE CUT TO, F12.8, 6H AT T, F12, 8)
C PRINT /5/*,N,T
GO TO 99
50 CONTINUE
99 RETURN
END

SUBLROUTINE TST (TSTRN)
C MONITORS FOR VM. END OF INTEG. OR PRINT RANGE.
COMMON//T,N,T,F,H,HO,H0,VM,VJ,ACC,XLB*RELST*ABSTST*FACTNO.ANN*TEST001
1X(30,5),F(30,5),E(30),XP(30),G(30,4),TDOUHL,NNURL
DIMENSION X(30),XP(30),F1(30),F2(30)
IF (ABS(E) .GE.1E-6) GO TO 20
IF ((X(M,J) .LE.VM) AND (Y(M,J-1) .LE.VM)) GO TO 10
IF ((X(M,J) .GT.VM) AND (Y(M,J-1) .GT.VM)) GO TO 10
GO TO 21
10 CALL RNGII
IPETPN=2
RETURN
20 IF (T.LT.TF) GO TO 40
30 IF (T.LT.TF) GO TO 40
35 X(I,J)=X(I,J-1)
J=2
CALL RNGII
IPETPN=2
RETURN
40 IF (T.LT.BND) GO TO 50
C SAVE ALL VARIABLES WHICH MAY BE MODIFIED IN PRINT PROCEDURE
HSAVE=H
TSAVE=T
SAVE=J
ON 45 I=1,N
X(I,J)=X(I,J-1)
XP(I,J)=XP(I,J-1)
F1(I,J)=F1(I,J-1)
F2(I,J)=F2(I,J-1)
45 X(I,J)=X(I,J-1)

28
SUBROUTINE DTONE
C FINISH VALUE OF T WHEN THE MTH VARIABLE REACHES THE VALUE VM
COMMON/F,TM,F,TF,VM,H0,H1,H2,H3,H4,H5,H6,H7,H8,H9,H10
C J=2
CALL RNGA
CALL PRINT(T,X(1:N))
RETURN
END

SUBROUTINE ADAMS
C INITIAL ONE STEP BY THE ADAMS PREDICTOR-CORRECTOR METHOD
COMMON/T,Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y9,Y10
C J=5
CALL DERIV(T,X(1:4),F(1:4))
IN 10 1=1,N
10 X(I)=X(I)+0.041666667*H*(59.0*F(I)+59.0*F(I+1))
IN 10 1=1,N
CALL DERIV(T,XP+F(1:5))
IN 10 1=1,N
20 X(I+1)=X(I)+0.041666667*H*(9.0*F(I)+9.0*F(I+1))
1=6.0*F(I)+9.0*F(I+1)
RETURN
END

SUBROUTINE NDOUBL (IRET)
C TEST TO SEE IF INTERVAL CAN BE DOUBLED
COMMON/Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y9,Y10
C J=10
CALL RNGA
CALL PRINT(T,X(1:N))
RETURN
END

TEST0045 TEST0044 TEST0047 TEST0044 TEST0049
TEST0050 TEST0051 TEST0052 TEST0053 TEST0054
TEST0055 TEST0056 TEST0057 TEST0058
TEST0059 TEST0060 TEST0061 TEST0062
TEST0063 TEST0064 TEST0065 TEST0066

DIODO010 DIODO011 DIODO012
DIODO013 DIODO014 DIODO015
DIODO016 DIODO017
DIODO018 DIODO019 DIODO020
DIODO021 DIODO022
DIODO023 DIODO024
DIODO025

ADAM0010 ADAM0011 ADAM0012
ADAM0013 ADAM0014 ADAM0015
ADAM0016 ADAM0017 ADAM0018
ADAM0019 ADAM0020 ADAM0021
ADAM0022 ADAM0023 ADAM0024
ADAM0025

DBLE0010 DBLE0011 DBLE0012
DBLE0013 DBLE0014

29
IF (INOURL.LT.NDUAL) GO TO 99

C ALLOWS NOUFL ATTEMPT ONLY EVERY NDUAL/TH CALL

INOURL=0

10 CONTINUE

IF (E(I).LT.XLAT) GO TO 99

IF (D1*E.RP0) GO TO 99

IF (D1*LF.RP0) GO TO 99

UN 20 I=1,N

20 X(I)=X(I+4)

15 CONTINUE

FORMAT (1D5,4F12.8)

RETURN

RETURN

END